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The effective action for the 4-point functions in abelian open superstring theory

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Abstract: We construct the derivative corrections to the four-point vertices in the abelian open string effective action to all orders in α'. The result is based on the structure of the string four-point function. Supersymmetry of these vertices is guaranteed by the supersymmetry of the F^4 term in the effective action. By this construction we establish the existence of an infinite number of supersymmetry invariants, the number of invariants at order α'^n grows linearly with n.

Keywords: Superstrings and Heterotic Strings, D-branes, Supersymmetric Effective Theories.

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Contents

1. Introduction

The problem of constructing the open superstring effective action in ten dimensions is still not settled. Even in absence of Chan-Paton factors (the abelian case) only a few sectors of the complete effective action are known. The ten-dimensional Born-Infeld action describes the dynamics for slowly varying fields [1], which in the abelian case is a consistent approximation. Its supersymmetric extension was derived in [2]–[5]. In [6] it was shown that there are no corrections quadratic in derivatives to all orders in α'. All bosonic terms with four derivatives were derived in [7]. Furthermore, it is known that there are no corrections with an odd number of fields strengths.[1]

In this paper we derive a new all-order result. We obtain the effective action for the tree-level four-point function in the abelian open superstring theory to all orders in α', i.e., including all derivative corrections. Our construction is an example of the so-called S-matrix method [8]–[11] to construct the effective action.[2] In this method one first writes down an action which reproduces the propagators of the massless string modes, and proceeds, in the absence of cubic interactions, to the four-point function, which in string theory is non-polynomial in the momenta $k_1, \ldots, k_4$ of the external particles. Because of the absence of cubic interactions the four-point function does not have poles, and the calculation of the four-point function only involves one-particle-irreducible diagrams.

[1] One can show that as a consequence of the invariance of the theory under worldsheet parity, all string amplitudes with an odd number of external lines (involving only massless modes) vanish [8]. The authors were unaware of this whilst writing [9], which led them to propose the above fact as a conjecture. This footnote should settle the issue.

[2] See [12] for a recent example of the use of the S-matrix method in the present context.

One can easily write down a closed form for the effective action because the open string four-point function factorizes in a product of two terms: the first term (K) depending on polarization vectors and wave functions, the second term ($\mathcal{G}$), proportional to the Veneziano amplitude, depending only on the momenta. The first term determines how the fields should appear in the effective action. The second term expands into an infinite series in α', and determines how derivatives should be distributed over the fields. This structure applies to both the bosonic terms and the terms involving fermions. Due to the factorization of the amplitude, supersymmetry of the effective action can be easily established. The supersymmetry of the effective action which reproduces the term K has been established a long time ago [13]. The term $\mathcal{G}$, with momenta replaced by derivatives acts on K in the full effective action, but we will show that the proof of supersymmetry still works "under the derivatives".

In discussing the higher derivative contributions to the open string effective action it is useful to introduce the following notation [9]. We write such terms as

$$\mathcal{L}_{(m,n)} = \alpha'^{\,m} \left(\partial^n F^p + \partial^{n+1} F^{p-2} \bar{\chi}\gamma\chi + \cdots \right). \tag{1.1}$$

For dimensional reasons we must have $2p - 2m + n - 4 = 0$.

The outline of this paper is as follows. In section 2 we review some properties of the tree-level four-point function in open string theory, and construct the corresponding bosonic effective action. We then proceed to discuss in section 3 the fermionic contributions and verify that the effective action is supersymmetric. In section 4 we consider the expansion of the result in α', and give explicit results through order $\alpha'^{\,5}$. Conclusions are given in section 5.

2. The 4-photon tree amplitude and its effective action

The open string tree-level 4-point function is given by [8]:

$$\mathcal{A}(1,2,3,4) = -16\, i\, g^2 \alpha'^{\,2} (2\pi)^{10} \delta^{(10)}(k_1 + k_2 + k_3 + k_4)\, \mathcal{G}(k_1,k_2,k_3,k_4)\, K(1,2,3,4) \tag{2.1}$$

$\mathcal{G}$ contains the α' dependence and is given by:

$$\begin{aligned} \mathcal{G}(k_1,k_2,k_3,k_4) &= G(s,t) + G(t,u) + G(u,s) \\ &= \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} + \frac{\Gamma(-\alpha' t)\Gamma(-\alpha' u)}{\Gamma(1-\alpha' t-\alpha' u)} + \frac{\Gamma(-\alpha' u)\Gamma(-\alpha' s)}{\Gamma(1-\alpha' u-\alpha' s)}\,. \end{aligned} \tag{2.2}$$

Here s, t, and u are the Mandelstam variables, satisfying $s + t + u = 0$. They are defined in terms of the k_i only up to momentum conservation and the mass-shell condition. We choose to write them in such a way that $\mathcal{G}$ is manifestly symmetric in the k_i:

$$\begin{aligned} s &= -\,k_1 \cdot k_2 - k_3 \cdot k_4\,, \\ t &= -\,k_1 \cdot k_3 - k_2 \cdot k_4\,, \\ u &= -\,k_1 \cdot k_4 - k_2 \cdot k_3\,. \end{aligned} \tag{2.3}$$

As discussed in the above, $\mathcal{G}$ is regular as $k_i \to 0$, which one can verify by expanding (2.2) in α'. For now we just mention that

$$\mathcal{G}(k_1, k_2, k_3, k_4) = -\frac{\pi^2}{2} + \mathcal{O}(\alpha'^2)\,, \tag{2.4}$$

and postpone a detailed discussion of the expansion to a later section. K involves not only the momenta of the external particles, but also their wavefunctions. For the 4-boson amplitude we have:

$$K(1,2,3,4) = t^{abcdefgh} k^1_a \zeta^1_b k^2_c \zeta^2_d k^3_e \zeta^3_f k^4_g \zeta^4_h\,, \tag{2.5}$$

where ζ^i is the polarization vector of the ith incoming photon, and the tensor t is defined in appendix A. The leading order contribution to the amplitude is just (2.5) times a constant. It is well known that it is reproduced by the action

$$\begin{aligned} S_{(2,0)} &= \frac{1}{8}(2\pi g\alpha')^2 \int \mathrm{d}^{10}x \left(\mathrm{tr}F^4 - \frac{1}{4}(\mathrm{tr}F^2)^2\right) \\ &= \frac{1}{8}(2\pi g\alpha')^2 \int \mathrm{d}^{10}x\, \frac{1}{24} t_{abcdefgh} F^{ab} F^{cd} F^{ef} F^{gh}\,, \end{aligned} \tag{2.6}$$

We observe that every factor of momentum k_i in (2.5) is reproduced by a derivative acting on the appropriate field in (2.6).

The complete amplitude (2.1) differs from the leading order contribution by multiplication with $\mathcal{G}$, i.e. by extra factors of momentum. In order to reproduce these factors, we simply need to act with derivatives on the appropriate fields. This is implemented by first allowing the four fields to be "defined at different points in spacetime", resulting in a non-local action. That is, we consider the fields $A_a(x_i)$, where $i = 1, \ldots, 4$, and then replace the momenta k_i in the amplitude by differentiations with respect to the appropriate coordinate in the effective action, i.e. $k_{i,a} \to -i\partial/\partial x^a_i$. We need to multiply the resulting expression by delta functions and then integrate over the x_i to make the action local.

Hence we define the following differential operator

$$D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}) \equiv \mathcal{G}(k_1, k_2, k_3, k_4)|_{k_i \to -i\partial_{x_i}}\,, \tag{2.7}$$

which we use to write down the effective action for the complete four-photon amplitude:

$$\begin{aligned} S_{\mathrm{eff}}[A_a] = &-\frac{1}{24}(g\alpha')^2 \int \mathrm{d}^{10}x \left\{\prod_i \mathrm{d}^{10}x_i\, \delta^{(10)}(x - x_i)\right\} D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}) \times \\ &\times t_{abcdefgh} F^{ab}(x_1) F^{cd}(x_2) F^{ef}(x_3) F^{gh}(x_4)\,. \end{aligned} \tag{2.8}$$

D is understood as a Taylor expansion in α'. Then the multiple integral over the x_i factorizes into a product of integrals, each involving only one of the x_i and none of the others, which is necessary in order that the above expression is well defined. The actual proof that this action reproduces the amplitude (2.1) can be found in appendix B.

As mentioned above, we choose to express s, t, u in terms of the k_i in such a way that $\mathcal{G}$ is manifestly symmetric in the momenta. This will turn out to be convenient in the following section. It is not difficult to see that a different prescription than (2.3) would result in modifications of the effective action (2.8) by total derivatives and/or the effects of field redefinitions.[3]. This follows from momentum conservation $k_1^a + k_2^a + k_3^a + k_4^a = 0$ and the mass-shell conditions $k_i^2 = 0$, respectively.

3. The fermionic contributions and supersymmetry

As is well known, the supersymmetric extension of (2.6) is unique and given by [13, 15]:[4]

$$S_{(2,0)} = \frac{1}{8}(2\pi g\alpha')^2 \int \mathrm{d}^{10}x \left(\mathrm{tr}F^4 - \frac{1}{4}(\mathrm{tr}F^2)^2 - 2F_{ab}F_{ac}\bar{\chi}\gamma_b\partial_c\chi + F_{ab}F_{cd}\bar{\chi}\gamma_{abc}\partial_d\chi + \right.$$
$$\left. + \frac{1}{3}\bar{\chi}\gamma_a\partial_b\chi\bar{\chi}\gamma_a\partial_b\chi \right) . \tag{3.1}$$

This action reproduces the four-point string amplitudes involving two and four fermions [16] to lowest order in α'. It is then easy to guess what the effective action should be when fermionic interactions as well as higher derivative corrections are included:

$$S_{\mathrm{eff}}[A_a, \chi] = -(g\alpha')^2 \int \mathrm{d}^{10}x \left\{ \prod_i \mathrm{d}^{10}x_i\, \delta^{(10)}(x - x_i) \right\} D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}) \times$$
$$\times \left\{ F_{ab}(x_1)F_{bc}(x_2)F_{cd}(x_3)F_{da}(x_4) - \frac{1}{4}F_{ab}(x_1)F_{ab}(x_2)F_{cd}(x_3)F_{cd}(x_4) - \right.$$
$$- 2F_{ab}(x_1)F_{ac}(x_2)\bar{\chi}(x_3)\gamma_b\partial_c\chi(x_4) + F_{ab}(x_1)F_{cd}(x_2)\bar{\chi}(x_3)\gamma_{abc}\partial_d\chi(x_4) +$$
$$\left. + \frac{1}{3}\bar{\chi}(x_1)\gamma_a\partial_b\chi(x_2)\bar{\chi}(x_3)\gamma_a\partial_b\chi(x_4) \right\} . \tag{3.2}$$

It is not difficult to prove that this action is supersymmetric. As explained in the previous section, the operator D is symmetric in the ∂_{x_i}. This implies that, when we apply the Noether method[5] to (3.2), we can perform the same manipulations as the ones necessary to demonstrate the supersymmetry of (3.1).

Consider for example the variation of the first term in (3.1). It is given by

$$\begin{aligned} \delta\left(\mathrm{tr}\,F^4\right) &= \delta F_{ab}F_{bc}F_{cd}F_{da} + F_{ab}\delta F_{bc}F_{cd}F_{da} + F_{ab}F_{bc}\delta F_{cd}F_{da} + F_{ab}F_{bc}F_{cd}\delta F_{da} \\ &= 4\, F_{ab}F_{bc}F_{cd}\delta F_{da} \, . \end{aligned} \tag{3.3}$$

The last step is of course completely trivial in the local case, but essential for proving the supersymmetry. In the non-local case (3.2), this last step is not automatic. We see that it is the symmetry of D that allows us to perform it.

[3]Remember that terms containing lowest order field equations can be induced in the effective action by means of a redefinition of the fields. See e.g. [14, 9].

[4]Supersymmetry holds only order by order in the number of fields, starting with the standard super-Maxwell action $F^2 + \bar{\chi}\gamma\partial\chi$, and requires modifications of the supersymmetry transformations at all orders. The superinvariants involving higher-derivative terms defined below have a similar structure.

[5]For a detailed description of the Noether method in the case of super Yang-Mills theory we refer to our previous papers [14, 9] with A. Collinucci.

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In addition to algebraic manipulations of the kind described above, it is also necessary to perform partial integrations to prove the supersymmetry. In the local case one encounters for example the following total derivative at an intermediate stage of the calculation:

$$\partial_a \left(F_{ab} \,\mathrm{tr}\, F^2 \,\bar{\epsilon}\gamma_b\chi \right) . \tag{3.4}$$

In the non-local case this term will manifest itself as

$$\left(\frac{\partial}{\partial x_1^a} + \frac{\partial}{\partial x_2^a} + \frac{\partial}{\partial x_3^a} + \frac{\partial}{\partial x_4^a} \right) F_{ab}(x_1) F_{cd}(x_2) F_{cd}(x_3) \,\bar{\epsilon}\gamma_b\chi(x_4) . \tag{3.5}$$

This still gives rise to a total derivative, since we can pull the $\sum_i \partial/\partial x_i^a$ out of the integration over the x_i:

$$\begin{aligned} \int \mathrm{d}^{10}x \left\{ \prod_i \mathrm{d}^{10}x_i \,\delta^{(10)}(x - x_i) \right\} D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}) \times \\ \times \left(\sum_j \frac{\partial}{\partial x_j^a} \right) F_{ab}(x_1) F_{cd}(x_2) F_{cd}(x_3) \,\bar{\epsilon}\gamma_b\chi(x_4) = \\ = \int \mathrm{d}^{10}x \frac{\partial}{\partial x} \int \left\{ \prod_i \mathrm{d}^{10}x_i \,\delta^{(10)}(x - x_i) \right\} D(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}) \times \\ \times F_{ab}(x_1) F_{cd}(x_2) F_{cd}(x_3) \,\bar{\epsilon}\gamma_b\chi(x_4) . \end{aligned} \tag{3.6}$$

Here the symmetry properties of D are not required.

We conclude, that the fact that (3.2) is supersymmetric follows immediately from the supersymmetry of (3.1).

The above actually shows that when we replace D in (3.2) by *any* symmetric differential operator $\Delta(\partial_{x_1}, \ldots, \partial_{x_4})$, we obtain a supersymmetric action.

4. Derivative expansion of the effective action

In this section we will consider the derivative expansion of the effective action (2.8). This will allow us to make contact with previously obtained results at order α'^4 as well as to present new results at order α'^5. But first let us discuss the form of the generic Lorentz invariant symmetric differential operator $\Delta(\partial_{x_1}, \ldots, \partial_{x_4})$ and determine the number of independent supersymmetric invariants that are possible at any given order in α'.

To find the form of $\Delta(\partial_{x_1}, \ldots, \partial_{x_4})$ we need the most general Lorentz invariant expression that is symmetric and regular in the momenta k_i, after which we substitute $k_i \to -i\partial_i$. In such an expression only combinations $k_i \cdot k_j$ and their products can enter.[6] Using momentum conservation and the mass-shell condition all such terms can be written as combinations of s, t, u. Any completely symmetric polynomial in s, t, u can be written as:

$$\sum_{k \le l \le m} \alpha'^{k+l+m} \, c_{k,l,m} \, \mathcal{P}(k, l, m) , \tag{4.1}$$

[6] We do not have to consider contractions with the ε-tensor, since all scalars that one can form by contracting it with the momenta k_i vanish.

where the $c_{k,l,m}$ are constants and

$$\mathcal{P}(k,l,m) = s^k t^l u^m + s^k t^m u^l + s^m t^k u^l + s^m t^l u^k + s^l t^m u^k + s^l t^k u^m \,. \tag{4.2}$$

Define

$$P(n) = s^n + t^n + u^n \,, \qquad Q = stu \,. \tag{4.3}$$

$\mathcal{P}(k,l,m)$ can be expressed in terms of $P(n)$ and Q:

$$\mathcal{P}(k,l,m) = Q^k \left(P(l-k)P(m-k) - P(l+m-2k) \right) . \tag{4.4}$$

Furthermore, it follows from $P(1)P(n-1) = 0$ that

$$P(n) = \frac{1}{2} P(2)P(n-2) + QP(n-3) \,. \tag{4.5}$$

We conclude that we can express (4.1) in powers of $P \equiv P(2)$ and Q:

$$\sum_{a,b} \alpha'^{2a+3b} \, d_{a,b} \, P^a Q^b \,, \tag{4.6}$$

where the $d_{a,b}$ are constants. The number $N_{P,Q}(m)$ of possible independent combinations of P and Q, at order α'^m in the above expansion, is given by

$$N_{P,Q}(m) = \begin{cases} \left[\frac{m}{6}\right] + 1 \,, & \text{if} \quad m \neq 6 \times \left[\frac{m}{6}\right] + 1 \\ \left[\frac{m}{6}\right] \,, & \text{if} \quad m = 6 \times \left[\frac{m}{6}\right] + 1 \,, \end{cases} \tag{4.7}$$

where $[x]$ denotes the largest integer smaller than x.

This implies that, for a given m, there are $N_{P,Q}(m)$ independent supersymmetric contributions to the open string tree-level effective action that contain terms of the form $\partial^{2m} F^4$.

We now turn to the derivative expansion of (3.2). We use the Taylor expansion for $\log \Gamma(1+z)$,

$$\log \Gamma(1+z) = -\gamma z + \sum_{m=2}^{\infty} (-1)^m \zeta(m) \frac{z^m}{m} \,, \tag{4.8}$$

where $\zeta(n)$ is the Riemann zeta-function, γ the Euler-Mascheroni constant, to obtain the following expression for $G(s,t)$:

$$\alpha'^2 \, G(s,t) = \frac{1}{st} \exp \left\{ \sum_{m=2}^{\infty} \alpha'^m \frac{\zeta(m)}{m} (s^m + t^m - (s+t)^m) \right\} . \tag{4.9}$$

This expression can be used to calculate the α' expansion of $\mathcal{G}(k_1, \ldots, k_4)$. We give here the first terms in this expansion, expressed in P and Q:

$$\begin{aligned} \mathcal{G}(k_1,\ldots,k_4) \,=\, & -\frac{1}{2}\pi^2 - \frac{1}{48}\alpha'^2 \pi^4 P - \frac{1}{2}\alpha'^3 \pi^2 \zeta(3)\, Q - \frac{1}{960}\alpha'^4 \pi^6 P^2 - \frac{1}{48}\alpha'^5 \pi^2 \,\times \\ & \times \big(\pi^2 \zeta(3) + 12\zeta(5)\big) PQ - \frac{1}{967680}\alpha'^6 \big(51\pi^8 P^3 + 8\pi^2 \big(31\pi^6 + 30240\zeta(3)^2\big) Q^2\big) \,- \end{aligned}$$

$$
-\frac{1}{960}\alpha'^7\,\pi^2\left(\pi^4\zeta(3)+10\,\pi^2\zeta(5)+120\,\zeta(7)\right)\,P^2Q-\frac{1}{58060800}\alpha'^8\times \tag{4.10}
$$
$$
\times\left(155\pi^{10}\,P^4+32\pi^2(67\pi^8+18900\,\pi^2\zeta(3)^2+453600\,\zeta(3)\zeta(5))\,PQ^2\right)-
$$
$$
-\frac{1}{967680}\alpha'^9\Bigg(\pi^2\left(51\pi^6\zeta(3)+504\,\pi^4\zeta(5)+5040\,\pi^2\zeta(7)+60480\,\zeta(9)\right)\times
$$
$$
\times\,P^3Q+8\,\pi^2(31\pi^6\zeta(3)+10080(\zeta(3)^3+2\zeta(9)))Q^3\Bigg)+\cdots.
$$

We see that, at least to this order, all possible combinations of P and Q indeed appear. String theory thus seems to make use of all available superinvariants. By substituting derivatives for momenta in the above expansion and inserting the resulting expression in (3.2), one can straightforwardly construct the contribution to the effective action at any desired order in α'. We demonstrate this for the bosonic terms at order α'^4 and α'^5. At order α'^4 we obtain:

$$
\begin{aligned}
\mathcal{L}_{(4,4)} &= \frac{1}{288}\pi^4\,g^2\alpha'^4\,t_{abcdefgh}\,\partial_kF_{ab}\partial_kF_{cd}\partial_lF_{ef}\partial_lF_{gh}\\
&= \frac{1}{36}\pi^4\,g^2\alpha'^4\Bigg((\partial_kF_{ab}\partial_lF_{bc}\partial_kF_{cd}\partial_lF_{da}+2\,\partial_kF_{ab}\partial_kF_{bc}\partial_lF_{cd}\partial_lF_{da})-\\
&\qquad -\frac{1}{4}\,(\partial_kF_{ab}\partial_kF_{ab}\partial_lF_{cd}\partial_lF_{cd}+2\,\partial_kF_{ab}\partial_lF_{ab}\partial_kF_{cd}\partial_lF_{cd})\Bigg)\,.
\end{aligned} \tag{4.11}
$$

This expression is consistent with results obtained previously by different methods [17]. We have also checked explicitly that the terms bilinear in the fermions - which were obtained in [9], see also [18] — are reproduced correctly. As always, this result is determined up to total derivatives and terms containing lowest order field equations.

At order α'^5 we obtain the following result:

$$
\begin{aligned}
\mathcal{L}_{(5,6)} &= -\frac{1}{6}\pi^2\zeta(3)\,g^2\alpha'^5\,t_{abcdefgh}\,\partial_k\partial_l\partial_mF_{ab}\partial_kF_{cd}\partial_lF_{ef}\partial_mF_{gh} \qquad (4.12)\\
&= -4\pi^2\zeta(3)\,g^2\alpha'^5\left(\partial_k\partial_l\partial_mF_{ab}\partial_kF_{bc}\partial_lF_{cd}\partial_mF_{da}-\frac{1}{4}\,\partial_k\partial_l\partial_mF_{ab}\partial_kF_{ab}\partial_lF_{cd}\partial_mF_{cd}\right).
\end{aligned}
$$

As was already mentioned above, each of the terms $\mathcal{L}_{(m,2m-4)}$ constructed in this paper are, together with the order α'^0 super-Maxwell action, supersymmetric to fourth order in the number of fields. From the point of view of the Noether procedure each of these terms contributes to genuine superinvariants that extend to all orders in the number of fields. One such superinvariant is the complete open superstring effective action, to which all $\mathcal{L}_{(m,2m-4)}$ contribute. One can then pose the question how many independent sub-invariants the string effective action contains. In [9] the general structure of the web of supersymmetric derivative corrections was discussed in some detail. It was argued there that the contributions which in the string effective action have coefficients involving powers of $\zeta(n)$, n odd, only, should form independent invariants.

The simplest assumption, which was posed as a conjecture in [9], is that the sectors $\mathcal{L}_{(2,0)}$, $\mathcal{L}_{(4,4)}$ and $\mathcal{L}_{(m,2m-4)}$, m odd, contain the next-to-leading-order contributions to

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separate superinvariants, and that there are no other all-order invariants starting at $\mathcal{L}_{(m,n)}$ for any m, n. The results obtained in the present paper do not falsify this conjecture.

Note that the conjecture implies that the terms involving, for example, $\zeta(9)P^3Q$ and $\zeta(9)Q^3$, which are independently invariant when supersymmetry to fourth order in the number of fields is considered, should become part of a single invariant if supersymmetry is required also at higher orders.

5. Summary and conclusions

We have obtained a new result to all orders in α' for a specific sector of the open superstring effective action: the four-point vertices. The bosonic four-derivative term agrees with [6], the fermionic contributions at that order agree with our result [9], which was obtained with the Noether procedure.

The bosonic part of the term at order ${\alpha'}^5$ (4.12) (six derivatives) can be compared with a conjecture by Wyllard [19].

In [19] it was conjectured that all derivative corrections to the Born-Infeld action follow from the corrections to the Wess-Zumino term. This conjecture is applied in [19] using the results for the Wess-Zumino term of [7] as input. We have taken the six-derivative corrections given in [19, formula (4.16)], and extracted the terms of fourth order in F. We find:

$$\begin{aligned}\mathcal{L}_{(5,6)\,\mathrm{Wyllard}} = -4\pi^2\zeta(3)\,g^2\alpha'^5\Big(&\partial_k\partial_l F_{ab}\partial_k\partial_m F_{bc}\partial_l\partial_m F_{cd}F_{da} - \\ &-\frac{1}{4}\,\partial_k\partial_l F_{ab}\partial_k\partial_m F_{ab}\partial_l\partial_m F_{cd}F_{cd}\Big)\,. \end{aligned} \tag{5.1}$$

This agrees, up to field redefinitions, with our result (4.12). However, this agreement should be interpreted with care. First of all the procedure of [19] involves an infinite series involving functional derivatives of the Born-Infeld action with respect to the field strength F. The conjecture requires an ordering prescription for these functional derivatives. For our comparison we have taken the simplest solution to this ordering ambiguity. Secondly, the corrections to the Wess-Zumino term in [7] are not complete. Other corrections, such as those evaluated in [20]–[22], will contribute as well. On applying Wyllard's proposal to these extra terms, further six-derivative corrections to the Born-Infeld term might be generated. Our agreement with [19] indicates that these extra terms do not give rise to new six-derivative F^4 terms in the Born-Infeld action.[7]

In figure 1 we show the present situation for the effective action of abelian open superstring. Black dots indicate sectors for which bosonic as well as fermionic terms are known, and supersymmetry has been established. The terms corresponding to the four-point function established in this paper are along the line $(m, 2m-4)$, where m is the order of α'. All bosonic four-derivative terms have been given in [7], but the fermionic contributions remain to be found. Clearly further progress requires a better understanding of the six- and higher-point functions from string theory. In the case of the four-point

[7] We are grateful to Niclas Wyllard for useful remarks and suggestions on these issues.

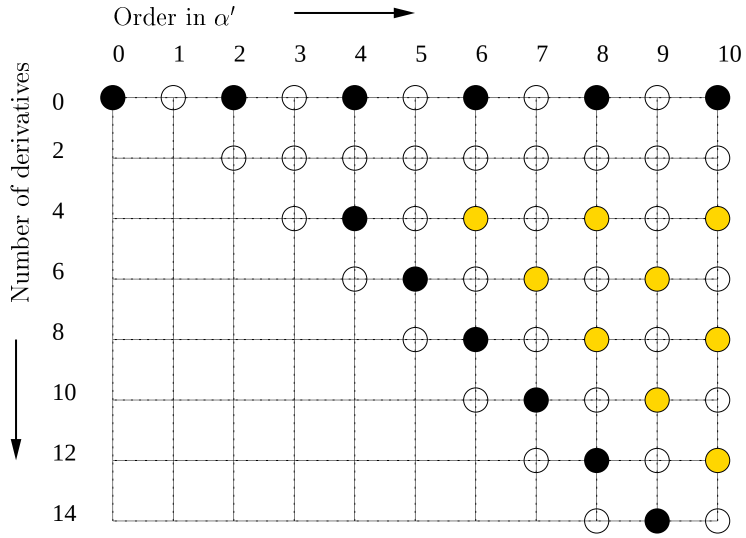

Figure 1: Structure of the abelian open superstring tree level effective action. Black dots indicate nonempty sectors of which the explicit form is known. Empty white dots correspond to sectors that are known to be empty up to field redefinitions. The yellow dots indicate sectors that are known to be nonempty but have yet to be constructed explicitly.

function supersymmetry of $\mathcal{L}_{(m,2m-4)}$ for $m > 0$ follows from the supersymmetry of $\mathcal{L}_{(2,0)}$. The generalization one could hope for is that supersymmetry of the full effective action follows, "under the derivatives", from supersymmetry of the Born-Infeld action.

An interesting problem is the extension of our result to the nonabelian case. In that case (2.1) is still valid, but $\mathcal{G}$ contains now also the group structure:

$$\begin{aligned}\mathcal{G}(k_1, k_2, k_3, k_4) = {} & (t_{ABCD} + t_{DCBA})G(s,t) + (t_{ABDC} + t_{CDBA})G(t,u) + \\ & + (t_{ACBD} + t_{DBCA})G(u,s)\,,\end{aligned} \tag{5.2}$$

where $t_{ABCD} = rmTr\lambda_A\lambda_B\lambda_C\lambda_D$. The problem is now that at order α'^n we are not just discussing the derivative correction to the four-point function, but also contributions with different numbers of derivatives and F's. These all communicate through the relation $[\mathcal{D}, \mathcal{D}]F = [F, F]$, and correspond to vertical lines in figure 1. The sectors which are independent in the abelian case are connected in the nonabelian situation. The method of [17] to organize the nonabelian effective string action in terms of symmetric traces seems to maximize the usefulness of the abelian results for solving the nonabelian problem. Nevertheless, making further progress with the nonabelian case remains a formidable problem.

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A. Definitions and conventions

We use the conventions of [9] for the metric, γ-matrices and fermions. We freely raise and lower spacetime indices. No confusion should arise, since contractions are always performed using the Minkowski metric.

An explicit expression for the tensor t_8 is given for example in [8]. $t_{abcdefgh}$ is antisymmetric in the pairs (ab), (cd), etc., and is symmetric under the exchange of such pairs. It satisfies the following identity:

$$\begin{aligned} t_{abcdefgh} M_1^{ab} M_2^{cd} M_3^{ef} M_4^{gh} = & -2\,(\mathrm{tr} M_1 M_2 \mathrm{tr} M_3 M_4 + \mathrm{tr} M_1 M_3 \mathrm{tr} M_2 M_4 + \mathrm{tr} M_1 M_4\, \mathrm{tr} M_2 M_3) + \\ & + 8\,(\mathrm{tr} M_1 M_2 M_3 M_4 + \mathrm{tr} M_1 M_3 M_2 M_4 + \mathrm{tr} M_1 M_3 M_4 M_2)\,, \end{aligned} \tag{A.1}$$

where the M_i are antisymmetric tensors.

The effective action is by definition the generator of 1PI diagrams:

$$S_{\mathrm{eff}}[A_a] \equiv \sum_n \frac{1}{n!} \int \mathrm{d}^{10}x_1 \cdots \mathrm{d}^{10}x_n\, \Gamma^{(n)}_{a_1 \cdots a_n}(x_1, \ldots, x_n) A^{a_1}(x_1) \cdots A^{a_n}(x_n)\,, \tag{A.2}$$

hence

$$\Gamma^{(n)}_{a_1 \cdots a_n}(x_1, \ldots, x_n) = \left. \frac{\delta^n S_{\mathrm{eff}}[A_a]}{\delta A^{a_1}(x_1) \ldots \delta A^{a_n}(x_n)} \right|_{A_a = 0} . \tag{A.3}$$

We define the momentum space amplitudes as follows:

$$(2\pi)^{10} \delta^{(10)}(k_1 + \ldots + k_n) \Gamma^{(n)}_{a_1 \cdots a_n}(k_1, \ldots, k_n) \equiv \int \prod_{i=1}^{n} \mathrm{d}^{10}x_i\, \mathrm{e}^{i k_i \cdot x_i}\, \Gamma^{(n)}_{a_1 \cdots a_n}(x_1, \ldots, x_n)\,. \tag{A.4}$$

An n-photon interaction gives the following contribution to the S-matrix:

$$\mathcal{A}(1, \ldots, n) = i (2\pi)^{10} \delta^{(10)}(k_1 + \ldots + k_n)\, \zeta^1_{a_1} \cdots \zeta^n_{a_n} \Gamma^{(n)}_{a_1 \cdots a_n}(k_1, \ldots, k_n)\,. \tag{A.5}$$

B. Proof

In order to reproduce (2.1), we have to obtain the following 1PI four-point function from (2.8):

$$\Gamma^{(4)}_{klmn}(k_1, k_2, k_3, k_4) = -16 (g\alpha')^2\, t_{akblcmdn} k_1^a k_2^b k_3^c k_4^d\, \mathcal{G}(k_1, k_2, k_3, k_4)\,. \tag{B.1}$$

First we calculate the four-point function in position space:

$$\begin{aligned} \Gamma^{(4)}_{klmn}(y_1, \ldots, y_4) &= \left. \frac{\delta^4 S_{\mathrm{eff}}[A_a]}{\delta A^k(y_1) A^l(y_2) A^m(y_3) A^n(y_4)} \right|_{A_a = 0} \\ &= -4! 2^4 \frac{1}{24} (g\alpha')^2 \int \mathrm{d}^{10}x \left\{ \prod_i \mathrm{d}^{10}x_i \delta(x - x_i) \right\} D(\partial_{x_1}, \ldots, \partial_{x_4}) t_{akblcmdn} \times \\ &\quad \times \partial^a_{x_1} \delta(x_1 - y_1) \partial^b_{x_2} \delta(x_2 - y_2) \partial^c_{x_3} \delta(x_3 - y_3) \partial^d_{x_4} \delta(x_4 - y_4)\,. \end{aligned} \tag{B.2}$$

The factor of 2^4 arises from substituting $F_{ab} = \partial_a A_b - \partial_b A_a$, the factor 4! from the distributive property of the functional derivative. To arrive at the result we renamed dummy variables and made use of the fact that D is symmetric in its arguments.

In momentum space this becomes:

$$
\begin{aligned}
&-\frac{1}{16(g\alpha')^2}(2\pi)^{10}\delta(k_1+k_2+k_3+k_4)\Gamma^{(4)}_{klmn}(k_1,k_2,k_3,k_4) = \\
&\quad = t_{akblcmdn}\int \mathrm{d}^{10}x \left\{\prod_i \mathrm{d}^{10}x_i \mathrm{d}^{10}y_i\, \delta(x_i-x)\mathrm{e}^{ik_i\cdot y_i}\right\} D(\partial_{x_1},\ldots,\partial_{x_4}) \times \\
&\qquad \times \partial^a_{x_1}\delta(x_1-y_1)\partial^b_{x_2}\delta(x_2-y_2)\partial^c_{x_3}\delta(x_3-y_3)\partial^d_{x_4}\delta(x_4-y_4) \\
&\quad = t_{akblcmdn}\int \mathrm{d}^{10}x \left\{\prod_i \mathrm{d}^{10}x_i\, \delta(x_i-x)\right\} \times \\
&\qquad \times D(\partial_{x_1},\ldots,\partial_{x_4})\, \partial^a_{x_1}\partial^b_{x_2}\partial^c_{x_3}\partial^d_{x_4} \left\{\prod_j \int \mathrm{d}^{10}y_j\, \mathrm{e}^{ik_j\cdot y_j}\delta(x_j-y_j)\right\} \\
&\quad = t_{akblcmdn}\int \mathrm{d}^{10}x \left\{\prod_i \mathrm{d}^{10}x_i\, \delta(x_i-x)\right\} \times \\
&\qquad \times \mathcal{G}(-i\partial_{x_1},\ldots,-i\partial_{x_4})\partial^a_{x_1}\partial^b_{x_2}\partial^c_{x_3}\partial^d_{x_4} \left\{\prod_j \mathrm{e}^{ik_j\cdot x_j}\right\} \\
&\quad = t_{akblcmdn}\int \mathrm{d}^{10}x \left\{\prod_i \mathrm{d}^{10}x_i\, \delta(x_i-x)\,\mathrm{e}^{ik_i\cdot x_i}\right\} \mathcal{G}(k_1,\ldots,k_4)\, k^a_1 k^b_2 k^c_3 k^d_4 \\
&\quad = t_{akblcmdn}\,\mathcal{G}(k_1,\ldots,k_4)\, k^a_1 k^b_2 k^c_3 k^d_4 \times (2\pi)^{10}\delta(k_1+k_2+k_3+k_4)\,.
\end{aligned}
\tag{B.3}
$$

This completes the proof.

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